On Verifying the First-Order Markovian Assumption for a Rayleigh Fading Channel Model

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***Abstract*— The use of received signal-to-noise ratio (SNR) as the side information in communication systems has been widely accepted especially when the channel quality is time varying. On many occasions, this side information is treated as the received SNR of the current channel symbol or that of previous symbols. In particular, the first-order Markov channel provides a mathematically tractable model for time-varying channels and uses only the received SNR of the symbol immediately preceding the current one. With the first-order Markovian assumption, given the information of the symbol immediately preceding the current one, any other previous symbol should be independent of the current one. Although the experimental measurements confirm the usefulness of the first-order Markovian assumption, one may argue that second or higher-order Markov processes should provide a more accurate model. In this paper, we answer this question by showing that given the information corresponding to the previous symbol, the amount of uncertainty remaining in the current symbol should be negligible.**

I. Introduction

In many communication systems, the noisy channel may possess certain time-varying memory content that causes channel quality to vary with time, depending on the previous channel condition. This phenomenon may cause unexpected degradation to the transmission because most subsystems are designed under the assumption of memoryless channel. To conquer this problem, interleavers are introduced, which can be used to mitigate the memory content of the channel if the length of the interleaver is sufficiently long. This, of course, introduces extra delay and complexity to the communication system. To turn the drawback of time-varying memory into an advantage, knowledge of previous channel conditions can be used to predict the upcoming channel quality and improve the performance of the communication system.

The use of received signal-to-noise ratio (SNR) as a measurement of channel quality in communication systems has been widely accepted especially when the channel quality is time-varying. On many occasions, this side information is treated as the received SNR of the current channel symbol [1], [2], [3] or that of previous symbols [4]. An example is referred to the finite-state Markov channel (FSMC) [5], a generalized Gilbert-Elliott channel [6], [7], [8] with a finite number of states. The FSMC model is constructed by partitioning the range of the received SNR into a finite number of intervals. Each state of the channel corresponds to one of these intervals. In this case, the first-order Markovian assumption implies that, given the information of the state immediately preceding the current one, any other previous state should be independent of the current state.

Generally speaking, the level of complexity incurred from using higher order Markov models can preclude a reasonable approach to parameterization of the model. In addition, accumulated errors in parameterization will result in questionable models, even if it can be done. These are the principal reasons higher-order Markov models are not considered. One may argue, however, that second- or higher-order Markov processes should provide a more accurate model even the experimental measurements and simulation results in [3], [5], [9] confirm the usefulness of the first-order Markovian assumption.

In this paper, we answer this question by showing that given the information corresponding to the previous symbol, the amount of uncertainty remaining in the current symbol should be negligible. Let $R_i, i \in \mathcal{R}$ be the received amplitude of the ith channel symbol, the information of R_i provided by the joint ensemble $R_{i-1}R_{i-2}$ can be quantified by the average mutual information $I(R_i; R_{i-1}R_{i-2})$ and decomposed as [10]

$$I(R_i; R_{i-1}R_{i-2}) = I(R_i; R_{i-1}) + I(R_i; R_{i-2} \mid R_{i-1}). \quad (1)$$

It is then clear that, given R_{i-1}, the significance of R_{i-2} in providing the information for R_i can be measured by the ratio of the *average conditional mutual information* $I(R_i; R_{i-2} \mid R_{i-1})$ and the *average mutual information* $I(R_i; R_{i-1}R_{i-2})$. Since the value of this ratio is a function of the joint probability density function of R_i, R_{i-1}, and R_{i-2}, it actually depends on the physical environment of the channel and the symbol transmission rate. In this paper, we use Rayleigh fading channels as an example with typical channel characteristics and transmission rates to verify the first-order Markovian assumption between consecutive channel symbols. The mathematical channel model is presented in Section II, followed by analytical derivation of the joint probability density function of the received envelope amplitudes in Section III. In Section IV, results from numerical evaluation and computer simulation are presented to show the relative importance of previous channel symbols to the current one. Conclusions are drawn in Section V.

II. Mathematical Model

We assume that a signal at frequency ω is transmitted. Following [11], waves arrive at the receiver with an angle $\alpha_n, n = 1, 2, \cdots, N$. At each of these N angles, there are M

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waves with propagation delay times T_{nm}. The electric field at the receiver can be written as

$$E(\omega,t) = E_0 \sum_{n=1}^{N}\sum_{m=1}^{M} C_{nm}\cos(\omega t + \omega_n t - \omega T_{nm}) \quad (2)$$

where $\omega_n = (2\pi v/\lambda)\cos\alpha_n$ is the Doppler shift resulting from the receiver's velocity v at the arrival angle α_n and λ is the wavelength of the transmitted carrier frequency. The amplitude coefficients C_{nm} are defined as

$$C_{nm}^2 = G(\alpha_n)p(\alpha_n, T_{nm})\, d\alpha dT \quad (3)$$

representing the power associated with each individual wave. With the approximated exponential distribution of the delay spreads and the assumption of a uniform distribution in angle of the incident power, the function $p(\alpha, T)$ can be written as

$$p(\alpha,T) = \frac{1}{2\pi T_D} e^{-T/T_D} \quad (4)$$

where T_D is a measure of the time delay spread. Furthermore, assuming that the antenna is omnidirectional, we have $G(\alpha) = 1$.

Equation (2) can be rewritten as

$$\underbrace{\left[E_0 \sum_{n=1}^{N}\sum_{m=1}^{M} C_{nm}\cos(\omega_n t - \omega T_{nm})\right]}_{x_I(t)}\cos\omega t - \underbrace{\left[E_0 \sum_{n=1}^{N}\sum_{m=1}^{M} C_{nm}\sin(\omega_n t - \omega T_{nm})\right]}_{x_Q(t)}\sin\omega t. \quad (5)$$

By the central limit theorem, $x_I(t)$ and $x_Q(t)$ are Gaussian random processes and are jointly Gaussian distributed for large numbers of N and M.

Since we are interested in the correlation of channel quality between three consecutive symbols, derivation of the joint probability density function (j.p.d.f.) of the corresponding envelope amplitudes is important. Let

$$\begin{aligned} x_1 &= x_I(t), \quad x_2 = x_I(t+\tau) \quad x_3 = x_I(t+2\tau), \\ x_4 &= x_Q(t), \quad x_5 = x_Q(t+\tau) \quad x_6 = x_Q(t+2\tau) \end{aligned} \quad (6)$$

then $x_1, \cdots, x_6$ are jointly Gaussian distributed and, without loss of generality, zero mean. Their second order statistics can be derived as

$$\langle x_1^2 \rangle = E_0^2 \sum_{n,m,p,q} \langle [C_{nm}C_{pq}\cos(\omega_n t - \omega T_{nm}) \times \cos(\omega_p t - \omega T_{pq})] \rangle \quad (7)$$

where $\langle\cdot\rangle$ denotes the ensemble average. Expanding (7) and noticing that the ensemble average is zero except for the terms with $n = p$ and $m = q$, let $\sigma^2 = E_0^2/2$, and substitute (3) and (4) into (7), then we have

$$\begin{aligned} \langle x_1^2 \rangle &= \sigma^2 \sum_{n=1}^{N}\sum_{m=1}^{M} G(\alpha_n)p(\alpha_n, T_{nm})\, d\alpha dT \\ &= \sigma^2 \int_0^\infty \int_0^{2\pi} G(\alpha)p(\alpha,T)\, d\alpha dT = \sigma^2. \end{aligned} \quad (8)$$

A similar argument leads to $\langle x_i^2 \rangle = \sigma^2$, $i = 1, \cdots, 6$. The ensemble average of x_1 and x_2 is

$$\begin{aligned} \langle x_1 x_2 \rangle &= E_0^2 \sum_{n,m,p,q} \langle C_{nm}C_{pq}\cos(\omega_n t - \omega T_{nm}) \\ &\qquad \times \cos(\omega_p(t+\tau) - \omega T_{pq}) \rangle \\ &= E_0^2 \sum_{n,m} C_{nm}^2 \Big\langle \cos^2(\omega_n t - \omega T_{nm})\cos(\omega_n\tau) \\ &\qquad - \frac{1}{2}\sin 2(\omega_n t - \omega T_{nm})\sin(\omega_n \tau)\Big\rangle. \end{aligned} \quad (9)$$

Since $\omega_n = (2\pi v/\lambda)\cos\alpha_n$, we have

$$\begin{aligned} \langle x_1 x_2 \rangle &= \sigma^2 \sum_{n,m} C_{nm}^2 \cos((2\pi v/\lambda)\tau\cos\alpha_n) \\ &= \sigma^2 \int_0^\infty \int_0^{2\pi} G(\alpha)p(\alpha,T) \\ &\quad \times \cos((2\pi v/\lambda)\tau\cos\alpha)\, d\alpha dT \\ &= \sigma^2 J_0(f_m\tau) \end{aligned} \quad (10)$$

where $J_0(\cdot)$ is the zero-order Bessel function and $f_m = (2\pi v/\lambda)$ is the maximum Doppler frequency. Similarly, we have

$$\langle x_1x_2\rangle = \langle x_2x_3\rangle = \langle x_4x_5\rangle = \langle x_5x_6\rangle = \sigma^2 J_0(f_m\tau), \quad (11)$$

$$\langle x_1x_3\rangle = \langle x_4x_6\rangle = \sigma^2 J_0(2f_m\tau) \quad (12)$$

and all the other cross correlations are zero. Finally, let $\underline{\mathbf{x}} = [x_1, \cdots, x_6]^t$, the joint probability density function of $\underline{\mathbf{x}}$ is

$$f(\underline{\mathbf{x}}) = \frac{1}{(2\pi)^3\sqrt{\det[\underline{\mathbf{C}}_x]}} \exp\left\{\frac{1}{2}\underline{\mathbf{x}}^t \underline{\mathbf{C}}_x^{-1}\underline{\mathbf{x}}\right\}. \quad (13)$$

With the j.p.d.f. of $\underline{\mathbf{x}}$, the j.p.d.f. of envelope amplitudes at t, $t+\tau$, and $t+2\tau$ can be derived.

III. Joint P.D.F. of Envelope Amplitudes

The relationship between $\underline{\mathbf{x}}$ and the envelope amplitudes and phases is

$$\begin{aligned} x_1 &= r_1\cos\theta_1, \quad x_2 = r_2\cos\theta_2, \quad x_3 = r_3\cos\theta_3 \\ x_4 &= r_1\sin\theta_1, \quad x_5 = r_2\sin\theta_2, \quad x_6 = r_3\sin\theta_3 \end{aligned} \quad (14)$$

where

$$r_1 = \sqrt{x_1^2 + x_4^2}, \quad r_2 = \sqrt{x_2^2 + x_5^2}, \quad r_3 = \sqrt{x_3^2 + x_6^2} \quad (15)$$

and

$$\theta_1 = \tan^{-1}\frac{x_4}{x_1}, \quad \theta_2 = \tan^{-1}\frac{x_5}{x_2}, \quad \theta_3 = \tan^{-1}\frac{x_6}{x_3}. \quad (16)$$

In order to derive the average conditional mutual information $I(R_i; R_{i-2} \mid R_{i-1})$, we need to first derive the joint probability density function $f_{R_1R_2R_3}(r_1, r_2, r_3)$. From (14), polar transformation can be performed to obtain the j.p.d.f. of r_i and θ_i, $i = 1, 2, 3$.

Let $J_1 = J_0(f_m\tau)$ and $J_2 = J_0(2f_m\tau)$, then

$$\underline{\mathbf{C}}_x = \sigma^2 \begin{bmatrix} \underline{\mathbf{A}} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{A}} \end{bmatrix}$$

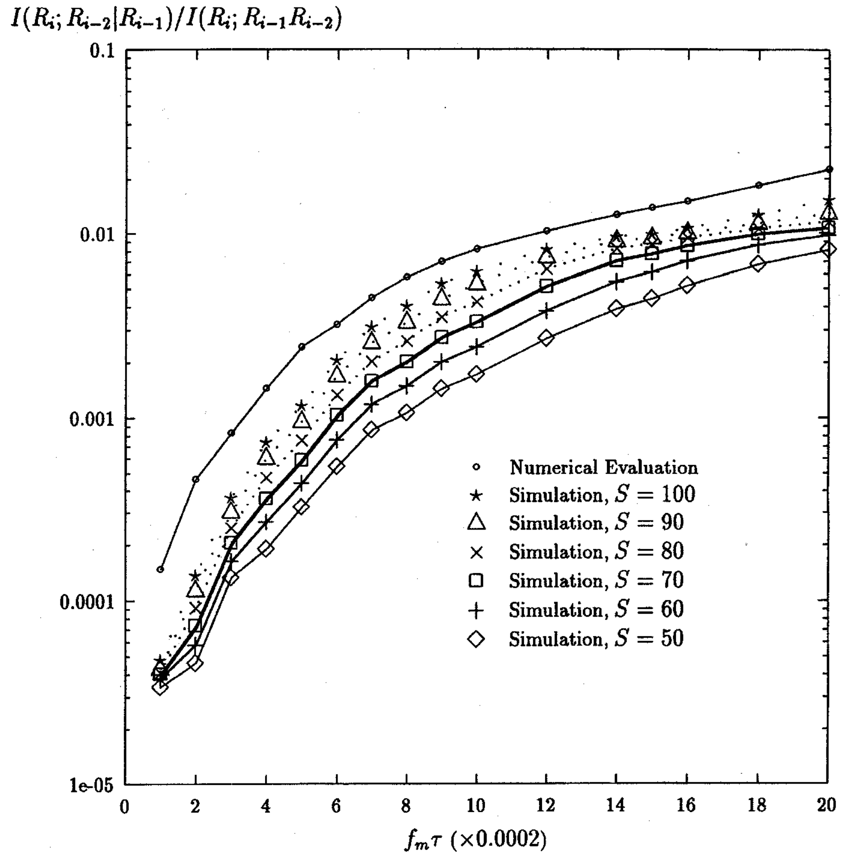

Fig. 1. Numerical evaluation and computer simulation results for the ratio between $I(R_i; R_{i-2} \mid R_{i-1})$ and $I(R_i; R_{i-1}R_{i-2})$.

where

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & J_1 & J_2 \\ J_1 & 1 & J_1 \\ J_2 & J_1 & 1 \end{bmatrix}.$$

Furthermore

$$\underline{\mathbf{C}}_x^{-1} = \frac{1}{\sigma^2}\begin{bmatrix} \underline{\mathbf{A}}^{-1} & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{A}}^{-1} \end{bmatrix}$$

where

$$\underline{\mathbf{A}}^{-1} = \begin{bmatrix} 1-J_1^2 & -J_1+J_1J_2 & J_1^2-J_2 \\ -J_1+J_1J_2 & 1-J_2^2 & -J_1+J_1J_2 \\ J_1^2-J_2 & -J_1+J_1J_2 & 1-J_1^2 \end{bmatrix}.$$

The polar transformation results in

$$\begin{aligned} f(r_1, r_2, r_3, \theta_1, \theta_2, \theta_3) &= \frac{r_1 r_2 r_3}{(2\pi)^3\sqrt{\det[\underline{\mathbf{C}}_x]}} \\ &\times \exp\bigg\{\frac{-1}{2}\big(\eta_{11}r_1^2 + \eta_{22}r_2^2 + \eta_{33}r_3^2 \\ &\quad + 2\eta_{12}r_1r_2\cos(\theta_1-\theta_2) \\ &\quad + 2\eta_{23}r_2r_3\cos(\theta_2-\theta_3) \\ &\quad + 2\eta_{31}r_3r_1\cos(\theta_3-\theta_1)\big)\bigg\} \end{aligned} \tag{17}$$

where η_{ij} is the i, jth element of $\underline{\mathbf{C}}_x^{-1}$. Finally, the j.p.d.f. of the envelope amplitudes can be obtained by integrating (17) with respect to θ_1, θ_2, and θ_3. After careful manipulation, we have

$$\begin{aligned} f_{R_1R_2R_3}&(r_1, r_2, r_3) \\ &= \frac{r_1r_2r_3}{\sigma^6\det[\underline{\mathbf{A}}]}\exp\bigg\{\frac{-1}{2}\big(\eta_{11}r_1^2 + \eta_{22}r_2^2 + \eta_{33}r_3^2\big)\bigg\} \\ &\quad \times \sum_{\Omega_{ijk}} \frac{(-\eta_{12}r_1r_2)^i(-\eta_{23}r_2r_3)^j(-\eta_{31}r_3r_1)^k}{i!j!k!}2^{-(i+j+k)} \\ &\quad \times \left[\sum_{l=l_{\min}}^{l_{\max}} \binom{i}{l}\binom{j}{l-\frac{i-j}{2}}\binom{k}{l-\frac{i-k}{2}}\right] \end{aligned} \tag{18}$$

where Ω_{ijk} is the set of nonnegative integers i, j, and k such that they are all even or all odd and $l_{\min}$ and $l_{\max}$ are defined as

$$l_{\min} = \max\left(0, \frac{i-j}{2}, \frac{i-k}{2}\right),$$

and

$$l_{\max} = \min\left(i, \frac{i+j}{2}, \frac{i+k}{2}\right).$$

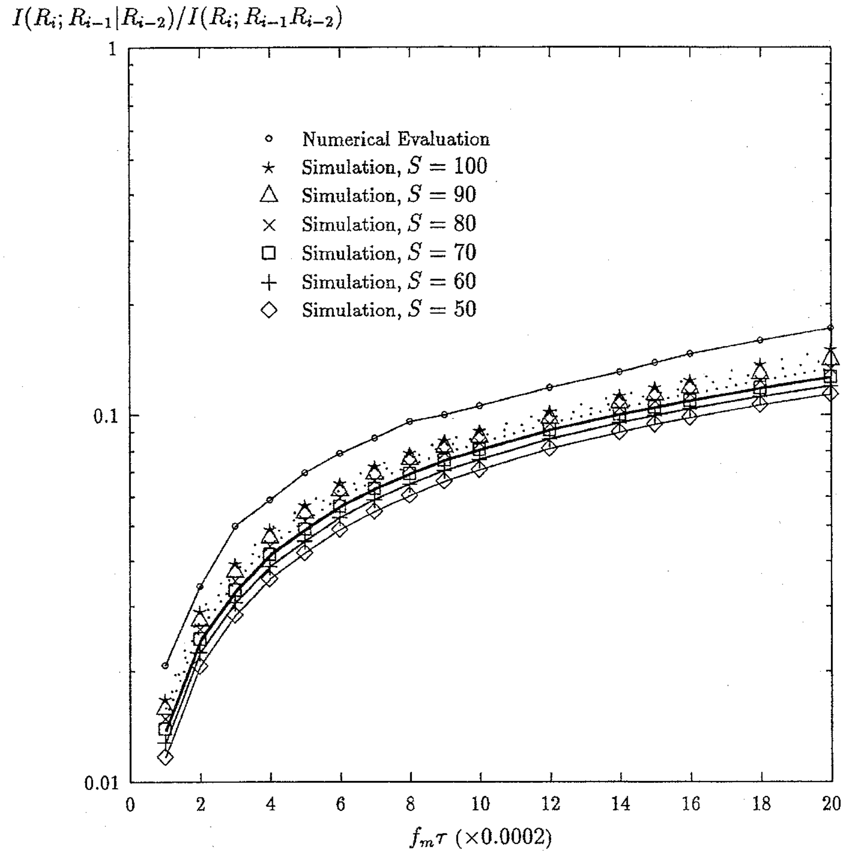

Fig. 2. Numerical evaluation and computer simulation results for the ratio between $I(R_i; R_{i-1} \mid R_{i-2})$ and $I(R_i; R_{i-1}R_{i-2})$.

IV. Numerical and Simulation Results

There are two important parameters in the model that characterize the fading speed. For a given value of f_m, the channel is considered to be fast fading if the duration between two channel symbols τ is large. This is because the two consecutive symbols are relatively uncorrelated for longer duration. On the other hand, if the channel symbol rate, or τ, is fixed, larger values of f_m imply fading with faster speed. The use of the product of f_m and τ as a measure for fast or slow fading is evidenced by (18) where the j.p.d.f. of received envelope amplitudes depends on f_m and τ through their product only.

Denote $I(R_i; R_{i-1}R_{i-2})$ as the mutual information between current channel symbol R_i and the ensemble of two immediately preceding symbols R_{i-1} and R_{i-2}. This quantity can be treated as the information about R_i that can be provided by R_{i-1} and R_{i-2}. As presented in (1), this quantity can be divided into two parts.

1) The information about R_i provided by the immediately preceding symbol, R_{i-1}.
2) The information about R_i provided by R_{i-2} given R_{i-1}.

What we are interested is how these two items are weighted. A straightforward measurement is the ratio between $I(R_i; R_{i-2} \mid R_{i-1})$ and $I(R_i; R_{i-1}R_{i-2})$.

In this section, we use two methods to evaluate the ratio of the average conditional mutual information $I(R_i; R_{i-2} \mid R_{i-1})$ and the average mutual information $I(R_i; R_{i-1}R_{i-2})$. Without a closed-form solution for the j.p.d.f. of R_1, R_2, and R_3, numerical evaluation has been carried out to calculate both

$$I(R_3; R_1 \mid R_2) = \iiint f(r_1, r_2, r_3) \log \frac{f(r_3, r_1 \mid r_2)}{f(r_1 \mid r_2) f(r_3 \mid r_2)} dr_1 dr_2 dr_3 \tag{19}$$

and

$$I(R_3; R_2R_1) = \iiint f(r_1, r_2, r_3) \log \frac{f(r_1, r_2, r_3)}{f(r_3) f(r_1, r_2)} dr_1 dr_2 dr_3. \tag{20}$$

In addition to the numerical result, we carry out a simulation of 120 s with various values of f_m and τ. The simulation results can approximate the ratio of the average mutual informations given in (19) and (20).

In our simulation, the number of waves M is equal to 50. The value of $f_m\tau$ has been chosen from 0.0002 to 0.0040, which are typical values encountered in practical Rayleigh fading channels. The range of received envelope is partitioned into S intervals. A sliding window of size three is used to approximate the joint probability mass functions (j.p.m.f) of (R_i, R_{i-1}, R_{i-2}), their pairwise j.p.m.f.'s, and their individual probability mass functions. These functions are then used to calculate the ratio of the average mutual informations mentioned above. In Fig. 1, simulation results with S ranging from 50 to 100 are presented along with the numerical result. It is noticed that the ratio between (19) and (20) is less than 1% for most values of $f_m\tau$, especially, when $f_m\tau$ is small. This is due to the fact that as the fading gets slower, the information about R_i has almost been fully explored given the knowledge of R_{i-1}. As the fading speed increases, the received signal envelope fluctuates rapidly and the information about current channel symbol that can be provided by previous symbols is limited. In general, the importance of R_{i-2} given R_{i-1} is negligible.

As a comparison, the roles of R_{i-1} and R_{i-2} are exchanged to investigate their relative importance. These results are shown in Fig. 2 where the ratio between $I(R_i; R_{i-1} \mid R_{i-2})$ and $I(R_i; R_{i-1}R_{i-2})$ are approximated. It is noticed that most of the results are from 0.01 to 0.1 and are about 10 to 100 times larger than that in Fig. 1.

V. Conclusion

In this paper, we derive the joint probability density function of the received envelope amplitudes at time $t, t+\tau$, and $t+2\tau$ under a Rayleigh fading environment. It has been observed that the fading speed depends on the maximum Doppler shift f_m and the duration between two consecutive channel symbols τ through their product. To investigate the correlation between the current channel symbol and the previous ones, the ratio between $I(R_i; R_{i-2} \mid R_{i-1})$ and $I(R_i; R_{i-1}R_{i-2})$ is approximated through numerical evaluation and computer simulation. We conclude that as for the current channel symbol, the effect of channel symbols other than the immediately preceding one is negligible. Therefore, the first-order Markovian assumption between channel symbols in a Rayleigh fading environment is verified.

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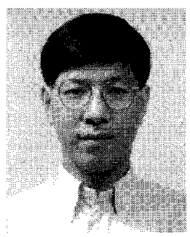

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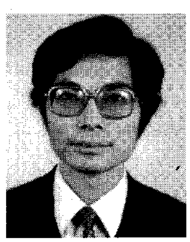

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